Received: *April 4, 2001*, Accepted: *May 22, 2001*

Properties of the Konishi multiplet in $\mathcal{N}=4$ SYM theory

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Abstract: We study perturbative and non-perturbative properties of the Konishi multiplet in $\mathcal{N}=4$ SYM theory in $D=4$ dimensions. We compute two-, three- and four-point Green functions with single and multiple insertions of the lowest component of the multiplet, $\mathcal{K}_{\mathbf{1}}$, and of the lowest component of the supercurrent multiplet, $\mathcal{Q}_{\mathbf{20'}}$. These computations require a proper definition of the renormalized operator, $\mathcal{K}_{\mathbf{1}}$, and lead to an independent derivation of its anomalous dimension. The $\mathrm{O}(g^2)$ value found in this way is in agreement with previous results. We also find that instanton contributions to the above correlators vanish. From our results we are able to identify some of the lowest dimensional gauge-invariant composite operators contributing to the OPE of the correlation functions we have computed. We thus confirm the existence of an operator belonging to the representation $\mathbf{20'}$, which has vanishing anomalous dimension at order g^2 and g^4 in perturbation theory as well as at the non-perturbative level, despite the fact that it does not obey any of the known shortening conditions.

Keywords: Nonperturbative Effects, 1/N Expansion, AdS-CFT Correspondance, Extended Supersymmetry.

JHEP05(2001)042

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JHEP05(2001)042

1. Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $D = 4$ is a very interesting and pedagogically useful theory. It is completely determined by the choice of the gauge group and is known to be "finite" [1]. All the couplings of the theory are related to the gauge coupling, which has vanishing β function both perturbatively and non-perturbatively. In the superconformal phase (at vanishing scalar *vev*'s), the spectrum of gauge invariant composite operators is very rich. They build up representations of the supergroup $\mathrm{SU}(2,2|4)$ that contains, as purely bosonic symmetries, the conformal group, $\mathrm{SO}(4,2)$, and the R-symmetry group, $\mathrm{SO}(6) \approx \mathrm{SU}(4)$.

The lowest (non-trivial) of such representations, the singleton, is made up by the 8+8 bosonic and fermionic components which are in correspondence with the fundamental fields of the theory. It has gauge invariant components only in the abelian case. In the non-abelian case, gauge invariant operators must be at least bilinears in the fundamental fields. The simplest gauge invariant supermultiplet is that of the

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$\mathcal{N} = 4$ supercurrent, which comprises 128 bosonic and as many fermionic operators. The "lowest lying" components of the multiplet are the so-called chiral primary operators (CPO's), $\mathcal{Q}_{\mathbf{20'}}$, belonging to the representation $\mathbf{20'}$ of SU(4)[1]. The operators $\mathcal{Q}_{\mathbf{20'}}$, as well as their superconformal partners (among which the stress-tensor, 4 supercurrents, 15 axial currents, ...), are protected against quantum corrections of their (conformal) dimensions, since the supercurrent multiplet is (ultra) short.

All multiplets built starting from CPO's with Dynkin labels of the form $[0, \ell, 0]$ and conformal dimension $\Delta_0 = \ell \geq 2$ are short multiplets. Relying on previous studies of unitary irreps of $SU(2, 2|\mathcal{N})$ [3], further multiplet shortening conditions have been identified in ref. [4] and classified in [5]. In particular the shortening of the multiplet built starting from the scalar operator of dimension $\Delta_0 = 4$ in the representation $\mathbf{84}$ of SU(4), which has Dynkin labels $[2, 0, 2]$,[2] nicely fits with the results obtained in [6, 7], where the precise field theoretical definition of this operator has been given. The problem with this and similar cases is that quantum corrections induce mixing among (bare) operators with the same quantum numbers and a basis of (independent) operators with well-defined conformal dimensions has to be identified. In the case at hand the relevant mixing involves operators of the so-called $\mathcal{N} = 4$ Konishi multiplet. The properties of its $\mathcal{N} = 1$ submultiplet (to which we will refer in the following as the $\mathcal{N} = 1$ Konishi multiplet) have been studied extensively after the discovery of the Konishi anomaly [8] (see e.g. [9] for a "pre–AdS/CFT" review). There has been a renewed interest in the subject in the light of the proposed AdS/CFT correspondence [10]–[13], because, as first observed in [14], the Konishi multiplet is a long multiplet that corresponds to the first string level in the spectrum of type IIB excitations around the $AdS_5 \times S^5$ background.

The purpose of this paper is to study the peculiar perturbative and non-perturbative properties of the $\mathcal{N} = 4$ Konishi multiplet. We will start by constructing the multiplet, taking in due account the terms induced by the Konishi anomaly. Following [14], we decompose the multiplet in terms of $\mathcal{N} = 1$ submultiplets. We will then compute correlation functions involving the lowest dimensional scalar operator of the multiplet, $\mathcal{K}_{\mathbf{1}}$, which is an SU(4) singlet with (naive) conformal dimension $\Delta_0 = 2$. Since it has a non-vanishing anomalous dimension, these computations require a careful definition of $\mathcal{K}_{\mathbf{1}}$, as a finite, gauge invariant, renormalized operator.

We will compute to $O(g^2)$ two-, three- and four-point correlators involving $\mathcal{K}_{\mathbf{1}}$ and/or the lowest scalar CPO's of the supercurrent multiplet. If complemented with what is known to $O(g^4)$ from refs. [7, 36], these calculations, besides confirming the results known both at $O(g^2)$ [15, 6] and at $O(g^4)$ [7] on the $\mathcal{K}_{\mathbf{1}}$ anomalous dimension, allow us to identify some of the operators contributing to the OPE of

[1]SU(4) representations are classified by a triplet of Dynkin labels $[k, \ell, m]$ [2]. In particular the Dynkin labels of the representation $\mathbf{20'}$ are $[0, 2, 0]$.

[2]Actually multiplets whose lowest scalar components have conformal dimension $\Delta_0 = \ell + 2k$ and Dynkin labels $[k, \ell, k]$ are short [4, 5].

these Green functions and compute their anomalous dimensions. This is done by exploiting certain rigorous bounds, that can be derived on general grounds on the $\mathrm{O}(g^2)$ corrections to the anomalous dimensions of operators contributing to the OPE of four-point Green functions. We are able to show in this way at $\mathrm{O}(g^4)$, as in [36], the vanishing of the anomalous dimension of an *a priori* unprotected scalar operator of dimension $\Delta_0 = 4$, belonging to the representation $\mathbf{20'}$, which was already shown to be zero at $\mathrm{O}(g^2)$ and non-pertubatively [16, 17].

We will then consider non-perturbative instantonic contributions to the same correlation functions and to our surprise we find vanishing results. This leads us to conclude that not only $\mathcal{K}_{\mathbf{1}}$ receives no instantonic contributions to its anomalous dimension, but it tends to display a much larger "inertia" to non-perturbative corrections. Our results imply the vanishing of non-perturbative corrections to all presently studied trilinear couplings involving components of the Konishi multiplet.

The plan of the paper is as follows. After fixing our notations in section 2 and giving the relevant components of the Konishi supermultiplet in section 3, in section 4 we construct the properly renormalized expression of $\mathcal{K}_{\mathbf{1}}$ and compute to $\mathrm{O}(g^2)$ its two-, three- and four-point Green functions. We also report explicit formulae for the four-point Green functions with single and multiple insertions of $\mathcal{K}_{\mathbf{1}}$ and of the lowest component operators belonging to the supercurrent multiplet, $\mathcal{Q}_{\mathbf{20'}}$. In section 5, we then derive a set of inequalities that anomalous dimensions of operators exchanged in intermediate channels must satisfy for consistency. With these results and the knowledge we have from the $\mathrm{O}(g^4)$ calculation of the Green functions of four $\mathcal{Q}_{\mathbf{20'}}$ operators, we show in section 6 what kind of information is possible to extract about the anomalous dimensions of the composite operators of naive dimension $\Delta_0 = 4$, belonging to the representation $\mathbf{20'}$ and to the singlet. An interesting corollary to this analysis is that we are able to extend to $\mathrm{O}(g^4)$, as in [36], the observation, made in ref. [16] to $\mathrm{O}(g^2)$, that there exists an operator in the representation $\mathbf{20'}$ which has vanishing anomalous dimension, despite the fact that it does not obey any of the known shortening conditions. At the same time we confirm the known results on the anomalous dimension of the Konishi multiplet. The instanton contributions to the Green functions considered in the previous sections are computed in section 7 and shown to vanish. Conclusions and an outlook of future lines of investigation can be found in section 8.

2. Notations and conventions

The field content of $\mathcal{N} = 4$ SYM [18] comprises a vector, A_μ, four Weyl spinors, ψ^A ($A = 1, 2, 3, 4$), and six real scalars, φ^i ($i = 1, 2, \ldots, 6$), all in the adjoint representation of the gauge group, $\mathrm{SU}(N)$.[3] In the $\mathcal{N} = 1$ approach that we shall follow

[3] The generalization to different gauge groups is straightforward.

the fundamental fields can be arranged into a vector superfield, V, and three chiral superfields, Φ^I $(I = 1, 2, 3)$. The six real scalars, φ^i, are combined into three complex fields, ϕ^I and $\phi^\dagger_I$, that are the lowest components of the chiral and antichiral superfields, Φ^I and $\Phi^\dagger_I$, respectively. Three of the Weyl fermions, ψ^I, are the spinors of the chiral multiplets. The fourth spinor, $\lambda = \psi^4$, together with the vector, A_μ, form the vector multiplet. In this way only an SU(3) $\otimes$ U(1) subgroup of the full SU(4) R-symmetry is manifest.

The complete $\mathcal{N} = 4$ SYM action in the $\mathcal{N} = 1$ superfield formulation has a non-polynomial form. A gauge fixing term must be added to the classical action. We shall use the Fermi-Feynman gauge, as it makes corrections to the propagators of the fundamental superfields vanish at order g^2 [1, 19, 20]. Actually a stronger result has been proved in these papers, namely the vanishing of the anomalous dimensions of the fundamental fields up to $\mathrm{O}(g^4)$. With the Fermi-Feynman gauge choice the terms relevant for the calculation of the Green functions we are interested in are

$$S = \int d^4x\, d^2\theta d^2\bar\theta \Big\{ V^a \Box V_a - \Phi^{a\dagger}_I \Phi^I_a - 2ig f_{abc} \Phi^{\dagger a}_I V^b \Phi^{Ic} + 2g^2 f_{abe} f_{ecd} \Phi^{\dagger a}_I V^b V^c \Phi^{Id} - \\ - \frac{ig\sqrt{2}}{3!} f^{abc} \Big[\varepsilon_{IJK} \Phi^I_a \Phi^J_b \Phi^K_c \delta^{(2)}(\bar\theta) - \varepsilon^{IJK} \Phi^\dagger_{aI} \Phi^\dagger_{bJ} \Phi^\dagger_{cK} \delta^{(2)}(\theta) \Big] + \cdots \Big\}, \tag{2.1}$$

where f_{abc} are the structure constants of the gauge group. As neither the cubic and quartic vector interactions nor the ghost terms will contribute to the calculations we will present in this paper, we have omitted them in eq. (2.1).[4]

Since all superfields are massless, their propagators have an equally simple form in momentum and in coordinate space and thus we choose to work in the latter, which is more suitable for the study of conformal field theories. In euclidean coordinate space one finds

$$\langle \Phi^\dagger_{Ia}(x_i, \theta_i, \bar\theta_i) \Phi^J_b(x_j, \theta_j, \bar\theta_j) \rangle = \frac{\delta_I{}^J \delta_{ab}}{4\pi^2} \mathrm{e}^{(\xi_{ii} + \xi_{jj} - 2\xi_{ji}) \cdot \partial_j} \frac{1}{x^2_{ij}}, \tag{2.2}$$

$$\langle V_a(x_i, \theta_i, \bar\theta_i) V_b(x_j, \theta_j, \bar\theta_j) \rangle = -\frac{\delta_{ab}}{8\pi^2} \frac{\delta^{(2)}(\theta_{ij}) \delta^{(2)}(\bar\theta_{ij})}{x^2_{ij}}, \tag{2.3}$$

where $x_{ij} = x_i - x_j$, $\theta_{ij} = \theta_i - \theta_j$, $\xi^\mu_{ij} = \theta^\alpha_i \sigma^\mu_{\alpha\dot\alpha} \bar\theta^{\dot\alpha}_j$.

3. The $\mathcal{N} = 4$ Konishi multiplet

The $\mathcal{N} = 4$ Konishi multiplet is a long multiplet of the superconformal group, SU(2, 2|4). Its lowest component, $\mathcal{K}_\mathbf{1}$, is a scalar operator of (naive) conformal di-

[4] Unlike what we have done in refs. [6] and [7], in this paper we use a more standard definition of g. To compare formulae of [6] and [7] with the present ones, one has to replace g there, with $2g$. We thank B. Eden and H. Osborn for pointing out to us that our notation was at odds with the standard one.

mension $\Delta_0 = 2$, which is a singlet of the SU(4) R-symmetry group. The highest spin component of the Konishi multiplet is a classically conserved spin 4 current, i.e. a 4 index totally symmetric tensor, of naive dimension $\Delta_0 = 6$, which is also a singlet of SU(4).

The (naive) definition of $\mathcal{K}_{\mathbf{1}}$ is

$$\mathcal{K}_{\mathbf{1}}(x)\Big|_{\text{naive}} = \frac{1}{2}\sum_{i=1}^{6} :\mathrm{tr}(\varphi^i(x)\varphi^i(x)):, \tag{3.1}$$

where the trace is over colour and the symbol : stands for normal ordering. As usual, normal ordering means subtracting the operator *vev* or, in other words, requiring $\langle\mathcal{K}_{\mathbf{1}}\rangle = 0$.

In terms of $\mathcal{N} = 1$ superfields the formal expression of $\mathcal{K}_{\mathbf{1}}$ is

$$\mathcal{K}_{\mathbf{1}}(x)\Big|_{\text{formal}} = \sum_{I=1}^{3} :\mathrm{tr}(\mathrm{e}^{-2gV(x,\theta,\bar\theta)}\Phi_I^\dagger(x,\theta,\bar\theta)\mathrm{e}^{2gV(x,\theta,\bar\theta)}\Phi^I(x,\theta,\bar\theta)):\Big|_{\theta=0,\bar\theta=0}, \tag{3.2}$$

where the exponents are included to ensure gauge invariance. From (3.2) it is clear that the $\mathcal{N} = 4$ Konishi multiplet contains the $\mathcal{N} = 1$ Konishi submultiplet. The latter is a real vector multiplet and among its components one finds the classically conserved U(1) axial current

$$\mathcal{K}_\mu = \bar\psi_A\bar\sigma_\mu\psi^A\,. \tag{3.3}$$

The anomalous divergence of the Konishi current is part of the Konishi anomaly [8], which in $\mathcal{N} = 1$ superfield notation reads

$$\frac{1}{4}\bar D^2 :\mathrm{tr}(\mathrm{e}^{-2gV}\Phi_I^\dagger\mathrm{e}^{2gV}\Phi^I) := \mathrm{tr}\left(\Phi^I\frac{\partial\mathcal{W}}{\partial\Phi^I}\right) + \frac{6g^2N}{32\pi^2}\mathrm{tr}(W^\alpha W_\alpha)\,, \tag{3.4}$$

where $\mathcal{W}$ is the superpotential, W^α is the chiral super-field-strength, defined as

$$W_\alpha = -\frac{1}{8g}\bar D^2(\mathrm{e}^{-2gV}D_\alpha\mathrm{e}^{2gV}) \tag{3.5}$$

and D, $\bar D$ are the $\mathcal{N} = 1$ supercovariant derivatives.

The presence of an anomalous divergence for the Konishi current (3.3) affects the expression of the superconformal descendants of $\mathcal{K}_{\mathbf{1}}$. In particular at level two (i.e. after acting with two supersymmetry transformations), the explicit form of the scalar operator in $\delta^2\mathcal{K}_{\mathbf{1}}$ reads

$$(\mathcal{K}_{\mathbf{10^*}})_{AB} = 3\sqrt{2}g t^{ijk}_{AB}\,\mathrm{tr}(\varphi_i\varphi_j\varphi_k) + \frac{6g^2N}{32\pi^2}\mathrm{tr}(\bar\psi_{\dot\alpha A}\bar\psi^{\dot\alpha}_B)\,, \tag{3.6}$$

where t^{ijk}_{AB} is the totally antisymmetric product of three matrices, t^i_{AB}, which in turn are the Clebsch-Gordan coefficients for the decomposition of the vector index i into two spinor indices A, B. The operator $(\mathcal{K}_{\mathbf{10}^*})_{AB}$ belongs to the representation $\mathbf{10}^*$ and bears a close resemblance to the operator

$$(\mathcal{E}_{\mathbf{10}^*})_{AB} = -\operatorname{tr}(\bar{\psi}_{\dot{\alpha}A}\bar{\psi}^{\dot{\alpha}}_{B}) + \sqrt{2}gt^{ijk}_{AB}\operatorname{tr}(\varphi_i\varphi_j\varphi_k)\,, \tag{3.7}$$

that is a superdescendant at level two of the chiral primary operator

$$\mathcal{Q}^{(ij)}_{\mathbf{20}'} = \operatorname{tr}\left(\varphi^i\varphi^j - \frac{\delta^{ij}}{6}\varphi^k\varphi^k\right). \tag{3.8}$$

$\mathcal{Q}_{\mathbf{20}'}$ belongs to the representation $\mathbf{20}'$ of SU(4) and, as we already said, is the lowest component of the $\mathcal{N}=4$ supercurrent multiplet.

The second, "anomalous", term in eq. (3.6) is obviously crucial in the construction of higher level operators and plays also a key rôle in making the two-point correlator $\langle\delta^2\mathcal{K}_{\mathbf{1}}(x)\delta^2\mathcal{Q}_{\mathbf{20}'}(y)\rangle$ vanish, i.e. in making $\mathcal{K}_{\mathbf{10}^*} = \delta^2\mathcal{K}_{\mathbf{1}}$ "orthogonal" to $\mathcal{E}_{\mathbf{10}} = \delta^2\mathcal{Q}_{\mathbf{20}'}$, as expected on the basis of the so-called $\mathrm{U}_B(1)$ "bonus symmetry" [21].

For later use, we need to get acquainted with the scalar operators of naive conformal dimension $\Delta_0 = 4$, that appear at level 4 in the Konishi multiplet. In order to construct $\delta^4\mathcal{K}_{\mathbf{1}}$, it is sufficient to perform two $\mathcal{N}=1$ supersymmetry transformations on

$$(\mathcal{K}_{\mathbf{6}^*\in\mathbf{10}^*})_{11} = 3\sqrt{2}g\ \operatorname{tr}(\phi_1^\dagger[\phi^2,\phi^3]) + \frac{6g^2N}{32\pi^2}\operatorname{tr}(\bar{\psi}_1\bar{\psi}_1)\,. \tag{3.9}$$

$(\mathcal{K}_{\mathbf{6}^*\in\mathbf{10}^*})_{11}$ is one of the components in the $\mathbf{6}^*$ of SU(3) which appears in the decomposition of the $\mathbf{10}^*$ of SU(4), to which the operator (3.6) belongs, with respect to the subgroup $\mathrm{SU}(3)\otimes\mathrm{U}(1)$. We are confident that naive supersymmetry transformations lead to the correct answer, since there is no need of normal-ordering the operator $(\mathcal{K}_{\mathbf{6}^*\in\mathbf{10}^*})_{11}$ that is used as a starting point. The final result is a complex scalar operator

$$\begin{aligned}(\mathcal{K}_{\mathbf{6}^*\in\mathbf{20}'})_{11} = {}& 6\sqrt{2}g\operatorname{tr}([\phi_1^\dagger,\psi^2]\psi^3) + 6\sqrt{2}g^2\left\{\operatorname{tr}([\phi_1^\dagger,\phi^2][\phi_2^\dagger,\phi_1^\dagger]) + \operatorname{tr}([\phi_1^\dagger,\phi_3^\dagger][\phi^3,\phi_1^\dagger])\right\} + \\ & + \frac{6g^2N}{32\pi^2}\left\{4\ \operatorname{tr}(D_\mu\phi_1^\dagger D^\mu\phi_1^\dagger) + \sqrt{2}g\ \operatorname{tr}([\phi_1^\dagger,\bar{\lambda}]\bar{\psi}_1^\dagger)\right\},\end{aligned} \tag{3.10}$$

that, as indicated, belongs to the $\mathbf{20}'$ of SU(4). In the following we will refer to it as $\mathcal{K}_{\mathbf{20}'}$. The conjugate operator, $\mathcal{K}^\dagger_{\mathbf{20}'}$, enters in the decomposition of $\bar{\delta}^4\mathcal{K}_{\mathbf{1}}$. It is important to stress that neither $\mathcal{K}_{\mathbf{20}'}$ nor $\mathcal{K}^\dagger_{\mathbf{20}'}$ appear in the OPE of $\mathcal{Q}_{\mathbf{20}'}\cdot\mathcal{K}_{\mathbf{1}}$ at the order we are going to work. The $\mathrm{U}_B(1)$ symmetry [21] would imply this to be true at any order in g^2.

The definition of the superdescendants in $\bar{\delta}^2\delta^2\mathcal{K}_{\mathbf{1}}$ is more involved. As far as the scalar components are concerned, one has an operator of naive dimension $\Delta_0 = 4$ in the $\mathbf{84}$ that can only involve scalar quadrilinear terms (terms involving fermionic fields would have conformal dimension larger than 4) and is consequently of the form

$$\mathcal{K}^{ij,kl}_{\mathbf{84}} = g^2\operatorname{tr}\left([\varphi^i,\varphi^j][\varphi^k,\varphi^l]\right). \tag{3.11}$$

At the same level there is an SU(4) singlet of the form

$$\mathcal{K}'_{\mathbf{1}} = 6\sqrt{2}g \left\{ t^i_{AB} \operatorname{tr}([\varphi_i, \psi^A]\psi^B) + t^{AB}_i \operatorname{tr}([\varphi^i, \bar{\psi}_A]\bar{\psi}_B) \right\} - 3g^2 \operatorname{tr}\left([\varphi^i, \varphi^j][\varphi_i, \varphi_j]\right) + \\ + \frac{6g^2 N}{32\pi^2} \left[2 \operatorname{tr}(D_\mu \varphi^i D^\mu \varphi_i) - \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}) \right] , \tag{3.12}$$

that may be thought of as the trace of the singlet classically conserved symmetric tensor, $\mathcal{K}_{\mu\nu}$, orthogonal to the exactly conserved and traceless canonical stress-energy tensor, $\mathcal{T}_{\mu\nu}$.

In addition to the components we have discussed, the $\mathcal{N} = 4$ Konishi multiplet contains a huge number (altogether 2^{16}) of other gauge invariant composite operators. A complete classification can be found in [14].

4. Correlation functions of the Konishi operator $\mathcal{K}_{\mathbf{1}}$

Let us consider the lowest dimensional operator, $\mathcal{K}_{\mathbf{1}}$, belonging to the Konishi multiplet. Its gauge invariant expression is given in eq. (3.2). As we said, $\mathcal{K}_{\mathbf{1}}$ has (naive) conformal dimension $\Delta_0 = 2$ and a non-vanishing anomalous dimension. Since we will only be interested in the $\theta = \bar{\theta} = 0$ component, effectively in the exponents of eq. (3.2) only the lowest component of the vector superfield, i.e. the scalar field $c(x)$, will be relevant. Similarly the chiral superfields $\Phi^I(x, \theta, \bar{\theta})$ ($\Phi^\dagger_I(x, \theta, \bar{\theta})$) will contribute only through their lowest components, $\phi^I(x)$ ($\phi^\dagger_I(x)$). Thus we have

$$\mathcal{K}_{\mathbf{1}}(x)\Big|_{\text{formal}} = \sum_{I=1}^{3} : \operatorname{tr}(\mathrm{e}^{-2gc(x)} \phi^\dagger_I(x) \mathrm{e}^{2gc(x)} \phi^I(x)) : \, . \tag{4.1}$$

In the Fermi-Feynman gauge that we use in this paper the propagator $\langle c(x)\, c(y) \rangle$ vanishes [20].

To give a precise meaning to the formal expression in eq. (4.1) one has to remember that, since the operator $\mathcal{K}_{\mathbf{1}}$ has an anomalous dimension, it will suffer a non-trivial renormalization. We find it convenient to regularize $\mathcal{K}_{\mathbf{1}}$ by point splitting

$$\mathcal{K}_{\mathbf{1}}(x)\Big|_{\text{reg}} = a(g^2) \sum_{I=1}^{3} : \operatorname{tr}\left(\mathrm{e}^{-2gc(x)} \phi^\dagger_I \left(x + \frac{\epsilon}{2}\right) \mathrm{e}^{2gc(x)} \phi^I \left(x - \frac{\epsilon}{2}\right)\right) : , \tag{4.2}$$

where ϵ is an infinitesimal, but otherwise arbitrary, four-vector. $a(g^2)$ is a normalization factor that admits an expansion of the form $a(g^2) = 1 + g^2 a_1 + g^4 a_2 + \cdots$. Unlike the operators corresponding to symmetry generators (like the R-symmetry currents or the stress-energy tensor), the Konishi scalar $\mathcal{K}_{\mathbf{1}}$ has no intrinsic normalization, so we shall use this freedom in the following to conveniently fix its normalization.

Note that, owing to our choice of gauge-fixing, there is no need to "point-split" the vector field in the exponents, because, as observed above, the c-field has vanishing propagator. An additional refinement to the formula (4.2) could be to make the replacement

$$\mathrm{e}^{gc(x)} \to \mathrm{e}^{g\int_{-1}^{1} d\rho c(x+\rho\epsilon/2)}\,, \tag{4.3}$$

but this would not affect the anomalous dimension of the operator and would only lead to a finite rescaling, which can be compensated by an appropriate change of the normalization constant, $a(g^2)$ in (4.2). So in the following we shall stick to the simpler regularized expression (4.2).

By conformal invariance, renormalization of $\mathcal{K}_{\mathbf{1}}$ simply amounts to a multiplication by the "renormalization constant"

$$Z(\mu\epsilon) = (\mu^2\epsilon^2)^{-\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)/2}\,, \tag{4.4}$$

where $\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)$ is the anomalous dimension of $\mathcal{K}_{\mathbf{1}}$ (by superconformal invariance $\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)$ is the anomalous dimension of the whole Konishi supermultiplet). We assume (as is always the case in perturbation theory) that $\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)$ is small and represented by the series expansion

$$\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2) = g^2\gamma_1 + g^4\gamma_2 + \cdots \tag{4.5}$$

From the results of refs. [15] and [6, 7] the first two coefficients of the expansion are known and (after the standard definition of g is used) are given by

$$\gamma_1 = \frac{3N}{4\pi^2} \tag{4.6}$$

$$\gamma_2 = -\frac{3N^2}{16\pi^4}\,. \tag{4.7}$$

Below we show that two-, three- and four-point Green functions of $\mathcal{K}_{\mathbf{1}}$ are made finite (up to order we have computed them), if the renormalized operator is taken to be of the form

$$\mathcal{K}_{\mathbf{1}}(x)\Big|_{\mathrm{renorm}} = \frac{a(g^2)}{(\epsilon^2)^{\frac{1}{2}\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)}} \sum_{I=1}^{3} :\mathrm{tr}\left(\mathrm{e}^{-2gc(x)}\phi_I^\dagger\left(x+\frac{\epsilon}{2}\right)\mathrm{e}^{2gc(x)}\phi^I\left(x-\frac{\epsilon}{2}\right)\right):\,, \tag{4.8}$$

provided the expansion (4.5) is used with γ_1 and γ_2 as in eqs. (4.6) and (4.7). In this equation and in the following, to avoid cumbersome formulae, we refrain from displaying explicitly the obvious μ dependence.

As a check of the correctness of eq. (4.8), one can compute the $\mathrm{O}(g^2)$ corrections to the two- and three-point functions of $\mathcal{K}_{\mathbf{1}}$. The coordinate dependence of these correlators is completely determined by conformal invariance. We shall use the freedom in the normalization factor $a(g^2)$ in eq. (4.8), to make the normalization of the two-point function independent of g^2. This is achieved at the order we work by setting $a_1 = 3N/8\pi^2$ (the other coefficients would require higher order computations to be fixed). With this choice one finds for the two-point function

$$\langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\rangle = \frac{3(N^2-1)}{4(4\pi^2)^2}\frac{1}{({x_{12}}^2)^{2+\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)}}\,, \tag{4.9}$$

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and for the three-point function

$$\langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\rangle = \frac{c_{\mathcal{KKK}}(g^2)}{({x_{12}}^2{x_{13}}^2{x_{23}}^2)^{1+\frac{1}{2}\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)}}\,, \tag{4.10}$$

where the cubic coupling is given by

$$c_{\mathcal{KKK}}(g^2) = \frac{3(N^2-1)}{4(4\pi^2)^3} + O(g^2)\,. \tag{4.11}$$

Taking $\gamma_1 = 3N/4\pi^2$, as in eq. (4.5), the g^2-expansion of the above expressions turns out to be in complete agreement with the results of the perturbative calculations, which give

$$\begin{aligned} \langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\rangle\Big|_{g^2} &= -\frac{9N(N^2-1)}{4(4\pi^2)^3}\frac{1}{x_{12}^4}\ln(x_{12}^2)\,, & (4.12)\\ \langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\rangle\Big|_{g^2} &= -\frac{9N(N^2-1)}{8(4\pi^2)^4}\frac{1}{x_{12}^2x_{13}^2x_{23}^2}[\ln(x_{12}^2x_{13}^2x_{23}^2)+3] & (4.13)\end{aligned}$$

for the $\mathrm{O}(g^2)$ corrections to two- and three-point correlators, respectively.

The tree-level value of the Green function of four Konishi scalars is

$$\begin{aligned} \langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{\mathrm{tree}} = \frac{(N^2-1)}{16(4\pi^2)^4}\Bigg[&\frac{9(N^2-1)}{x_{12}^4x_{34}^4} + \frac{9(N^2-1)}{x_{14}^4x_{23}^4} + \\ &+\frac{9(N^2-1)}{x_{13}^4x_{24}^4} + \frac{6}{x_{12}^2x_{34}^2x_{13}^2x_{24}^2} + \\ &+\frac{6}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2} + \frac{6}{x_{14}^2x_{23}^2x_{13}^2x_{24}^2}\Bigg]\,, \end{aligned} \tag{4.14}$$

while for the $\mathrm{O}(g^2)$ correction we find

$$\begin{aligned} \langle \mathcal{K}_{\mathbf{1}}(x_1)\mathcal{K}_{\mathbf{1}}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{g^2} = -&\frac{3N(N^2-1)}{16(4\pi^2)^5}\Bigg[\frac{B(r,s)}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2}\times \\ &\times(1+r^2+s^2+4r+4s+4rs) + \\ &+\frac{6}{x_{12}^2x_{34}^2x_{13}^2x_{24}^2}\left(\ln\left(\frac{x_{12}^2x_{34}^2x_{13}^2x_{24}^2}{x_{14}^2x_{23}^2}\right)+2\right) + \\ &+\frac{6}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2}\left(\ln\left(\frac{x_{12}^2x_{34}^2x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\right)+2\right) + \\ &+\frac{6}{x_{14}^2x_{23}^2x_{13}^2x_{24}^2}\left(\ln\left(\frac{x_{14}^2x_{23}^2x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2}\right)+2\right) + \\ &+\frac{9(N^2-1)}{x_{12}^4x_{34}^4}\ln\left(x_{12}^2x_{34}^2\right) + \frac{9(N^2-1)}{x_{14}^4x_{23}^4}\ln\left(x_{14}^2x_{23}^2\right) + \\ &+\frac{9(N^2-1)}{x_{13}^4x_{24}^4}\ln\left(x_{13}^2x_{24}^2\right)\Bigg]\,, \end{aligned} \tag{4.15}$$

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where the massless scalar box integral

$$B(r,s) = \frac{1}{\sqrt{p}}\Bigg\{\ln(r)\ln(s) - \left[\ln\left(\frac{r+s-1-\sqrt{p}}{2}\right)\right]^2 - \\ - 2\mathrm{Li}_2\left(\frac{2}{1+r-s+\sqrt{p}}\right) - 2\mathrm{Li}_2\left(\frac{2}{1-r+s+\sqrt{p}}\right)\Bigg\}, \quad (4.16)$$

is a function of the two conformally invariant ratios

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad s = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (4.17)$$

In eq. (4.16) we have introduced the definition

$$p = 1 + r^2 + s^2 - 2r - 2s - 2rs \quad (4.18)$$

and assumed proper analytic continuation of $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$.

Note that, contrary to the case of the four-point functions involving only protected operators, which depend only on the conformally invariant cross ratios (4.17), the four-point function (4.15) has an explicit logarithmic dependence on coordinate differences. This behaviour, which is similar to what one finds for the two- and three-point functions (4.12) and (4.13), does not contradict conformal invariance and it is just a manifestation of the anomalous dimension of the Konishi scalar $\mathcal{K}_{\mathbf{1}}$.

Let us now consider Green functions involving both the protected operators $\mathcal{Q}_{\mathbf{20'}}$ in (3.8) and the unprotected operator $\mathcal{K}_{\mathbf{1}}$.

We recall that the case of the four protected operators $\mathcal{Q}_{\mathbf{20'}}^{(ij)}$ has been studied previously. The relevant correlators are known explicitly up to $\mathrm{O}(g^4)$ and at the one-instanton level [6, 7, 22, 23].

In terms of $\mathrm{SU}(3)\otimes\mathrm{U}(1)$ the $\mathcal{Q}_{\mathbf{20'}}^{(ij)}$'s decompose in

$$\mathcal{C}^{IJ}(x) = \mathrm{tr}(\phi^I(x)\phi^J(x)), \qquad \mathcal{C}^{\dagger}_{IJ}(x) = \mathrm{tr}(\phi^{\dagger}_I(x)\phi^{\dagger}_J(x)) \quad (4.19)$$

and

$$\mathcal{V}^I_J = \mathrm{tr}\left(\mathrm{e}^{-2gc(x)}\phi^{\dagger}_J(x)\mathrm{e}^{2gc(x)}\phi^I(x)\right) - \frac{\delta^I_J}{3}\,\mathrm{tr}\left(\mathrm{e}^{-2gc(x)}\phi^{\dagger}_L(x)\mathrm{e}^{2gc(x)}\phi^L(x)\right), \quad (4.20)$$

where again the operators have been regularized by point-splitting like in eq. (4.2). Note that no normal-ordering is needed as the *vev*'s of all these operators vanish, since they are not SU(4) singlets.

As an additional check of the correctness of our approach, we may compute the three-point function of two protected operators $\mathcal{Q}_{\mathbf{20'}}$ and one $\mathcal{K}_{\mathbf{1}}$, for which again we find a perturbative expression in perfect agreement with the form required by conformal invariance, which reads

$$\langle \mathcal{Q}_{\mathbf{20'}}^{(i_1j_1)}(x_1)\mathcal{Q}_{\mathbf{20'}}^{(i_2j_2)}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\rangle = \frac{c_{\mathcal{Q}^{(i_1,j_1)}\mathcal{Q}^{(i_2j_2)}\mathcal{K}}(g^2)}{{x_{12}}^2{x_{13}}^2{x_{23}}^2}\left(\frac{{x_{12}}^2}{{x_{13}}^2{x_{23}}^2}\right)^{\frac{1}{2}\gamma_{\mathcal{K}_{\mathbf{1}}}(g^2)}. \quad (4.21)$$

Indeed one finds at tree-level

$$\langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\rangle\Big|_{\rm tree} = \frac{(N^2-1)}{2(4\pi^2)^3}\frac{1}{x_{12}^2 x_{13}^2 x_{23}^2} \tag{4.22}$$

and at $\mathrm{O}(g^2)$

$$\langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\rangle\Big|_{g^2} = \frac{3N(N^2-1)}{4(4\pi^2)^4}\frac{1}{x_{12}^2 x_{13}^2 x_{23}^2}\left(\ln\left(\frac{x_{12}^2}{x_{13}^2 x_{23}^2}\right)-1\right). \tag{4.23}$$

Again this result is consistent with the known $\mathrm{O}(g^2)$ value of the $\mathcal{K}_{\mathbf{1}}$ anomalous dimension.

Passing to four-point Green functions, there are only two non-vanishing choices

$$\langle \mathcal{Q}^{(i_1j_1)}_{\mathbf{20'}}(x_1)\mathcal{Q}^{(i_2j_2)}_{\mathbf{20'}}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle \tag{4.24}$$
$$\langle \mathcal{Q}^{(i_1j_1)}_{\mathbf{20'}}(x_1)\mathcal{Q}^{(i_2j_2)}_{\mathbf{20'}}(x_2)\mathcal{Q}^{(i_3j_3)}_{\mathbf{20'}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\,, \tag{4.25}$$

because the expectation value of three $\mathcal{K}_{\mathbf{1}}$ with one $\mathcal{Q}^{(ij)}_{\mathbf{20'}}$ is trivially zero due to SU(4) symmetry.

To avoid cumbersome notations, instead of writing correlators for generic SU(4) labels, we shall choose representatives. Note that reconstructing the general expression of the amplitudes is immediate, since in the cases of interest there is only one independent SU(4) structure for both the Green functions (4.24) and (4.25). In fact in the corresponding SU(4) tensor product there is only one singlet, unlike what happens for the $\mathcal{Q}_{\mathbf{20'}}\mathcal{Q}_{\mathbf{20'}}\mathcal{Q}_{\mathbf{20'}}\mathcal{Q}_{\mathbf{20'}}$ product, where there are six singlets (related by permutations and by a non-trivial functional relation [7, 23]). In particular we find for the Green function with two $\mathcal{Q}_{\mathbf{20'}}$'s and two $\mathcal{K}_{\mathbf{1}}$'s at tree-level

$$\begin{aligned}\langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{\rm tree} &= \\ = \frac{(N^2-1)}{4(4\pi^2)^4}\Bigg[\frac{1}{x_{12}^2x_{34}^2x_{13}^2x_{24}^2} + \frac{1}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2} &+ \frac{1}{x_{14}^2x_{23}^2x_{13}^2x_{24}^2} + \frac{3(N^2-1)}{2x_{12}^4x_{34}^4}\Bigg],\end{aligned} \tag{4.26}$$

while the $\mathrm{O}(g^2)$ correction is

$$\begin{aligned}\langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{K}_{\mathbf{1}}(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{g^2} = & -\frac{N(N^2-1)}{8(4\pi^2)^5}\Bigg[\frac{B(r,s)}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2}\times \\ & \times(1+r^2+s^2-2r+4s-2rs)+ \\ & +\frac{3}{x_{12}^2x_{34}^2x_{13}^2x_{24}^2}\left(\ln\left(\frac{x_{34}^4x_{13}^2x_{24}^2}{x_{14}^2x_{23}^2}\right)+2\right)+ \\ & +\frac{3}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2}\left(\ln\left(\frac{x_{34}^4x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\right)+2\right)+ \\ & +\frac{3}{x_{14}^2x_{23}^2x_{13}^2x_{24}^2}\left(\ln\left(\frac{x_{14}^2x_{23}^2x_{13}^2x_{24}^2}{x_{12}^4}\right)+2\right)+ \\ & +\frac{9(N^2-1)}{x_{12}^4x_{34}^4}\ln\left(x_{34}^2\right)\Bigg].\end{aligned} \tag{4.27}$$

This function is of particular interest, since in the 3-4- and 2-3-channels ($x_{34} \to 0$ and $x_{23} \to 0$, respectively) it gets contributions from operators in well defined SU(4) representations. Indeed only the singlet can contribute in the 3-4-channel and only the representation $\mathbf{20'}$ in the 2-3-channel.

For the Green function with three $\mathcal{Q}_{\mathbf{20'}}$'s and one $\mathcal{K}_{\mathbf{1}}$, we find at tree-level

$$\begin{aligned} &\langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{V}^2_2(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{\text{tree}} = \\ &= -\frac{(N^2-1)}{12(4\pi^2)^4}\left[\frac{1}{x^2_{12}x^2_{34}x^2_{13}x^2_{24}} + \frac{1}{x^2_{12}x^2_{34}x^2_{14}x^2_{23}} + \frac{1}{x^2_{14}x^2_{23}x^2_{13}x^2_{24}}\right] \end{aligned} \tag{4.28}$$

and at $\mathrm{O}(g^2)$

$$\begin{aligned} \langle \mathcal{C}^{11}(x_1)\mathcal{C}^{\dagger}_{11}(x_2)\mathcal{V}^2_2(x_3)\mathcal{K}_{\mathbf{1}}(x_4)\rangle\Big|_{g^2} = \frac{N(N^2-1)}{24(4\pi^2)^5}\Bigg[&\frac{B(r,s)}{x^2_{12}x^2_{34}x^2_{14}x^2_{23}} \times \\ &\times(1+r^2+s^2-2r-2s-2rs) - \\ &-\frac{3}{x^2_{12}x^2_{34}x^2_{13}x^2_{24}}\left(\ln\left(\frac{x^2_{23}}{x^2_{24}x^2_{34}}\right)-1\right) - \\ &-\frac{3}{x^2_{12}x^2_{34}x^2_{14}x^2_{23}}\left(\ln\left(\frac{x^2_{13}}{x^2_{14}x^2_{34}}\right)-1\right) - \\ &-\frac{3}{x^2_{14}x^2_{23}x^2_{13}x^2_{24}}\left(\ln\left(\frac{x^2_{12}}{x^2_{14}x^2_{24}}\right)-1\right)\Bigg]. \end{aligned} \tag{4.29}$$

In this case in all channels only the representation $\mathbf{20'}$ can be exchanged.

5. Anomalous dimensions and OPE

In this section we analyze the general structure of the consistency conditions that can be obtained combining conformal invariance, which requires power-like expressions for the Green functions, with the observed logarithmic behaviour found in perturbation theory. In particular an exact resummation of the logarithms to all orders in g^2 is expected to take place. Although this has been explicitly checked so far only at order $\mathrm{O}(g^4)$ for the lowest operator of the supercurrent multiplet, we shall assume it to be true in general, since otherwise scale invariance would be violated.

We shall consider a general four-point function of not necessarily protected operators. We schematically write it in the form $\langle Q_1(x_1)Q_2(x_2)Q_3(x_3)Q_4(x_4)\rangle$, and work in the double OPE limit, say, $x_1 \to x_2$ and simultaneously $x_3 \to x_4$ (s-channel). In this limit the perturbative corrections to the four-point functions exhibit a logarithmic behaviour, which, as we said, is a manifestation of the non-vanishing anomalous dimensions of the external operators and/or of the operators exchanged in the intermediate channel. For simplicity we shall concentrate only on the leading logarithmic contributions, that behave like $\ln^n(x^2_{ij})$ at order $\mathrm{O}(g^{2n})$, but similar considerations are valid also for subleading logarithmic terms.

Let us consider the logarithmic contributions coming from a finite set of operators O_i, $i = 1, \ldots, k$, all with the same tree-level scale dimension, Δ_0, but with different $\mathrm{O}(g^2)$ anomalous dimensions. Note that operators with the same anomalous dimension cannot be separated unambiguously in this approach. We assume that the intermediate operators have been made orthogonal at tree-level in the sense that they are chosen to satisfy the equations

$$\langle O_i(x_i) O_j(x_j) \rangle \Big|_{\mathrm{tree}} = \delta_{ij} \frac{N_i}{(x_{ij}^2)^{\Delta_0}} , \tag{5.1}$$

with $N_i > 0$ due to positivity.

The coefficients of the leading logarithm $\ln^n(x_{24}^2)$ (due to conformal invariance, consideration of the other two leading logarithmic behaviours, i.e. $\ln^n(x_{12}^2)$ and $\ln^n(x_{34}^2)$ does not lead to independent conditions) will satisfy the following relations

- tree-level (corresponding to $n = 0$)

$$\sum_{i=1}^{k} F_i = P_0 , \tag{5.2}$$

- order $\mathrm{O}(g^2)$ (corresponding to $n = 1$)

$$\sum_{i=1}^{k} F_i \gamma_i = P_1 , \tag{5.3}$$

- order $\mathrm{O}(g^4)$ (corresponding to $n = 2$)

$$\sum_{i=1}^{k} F_i (\gamma_i)^2 = P_2 , \tag{5.4}$$

where γ_i is the $\mathrm{O}(g^2)$ correction to the anomalous dimension of the i-th operator, O_i, and the coefficients, F_i, are ratios of tree-level normalization constants

$$F_i = \frac{c_{Q_1 Q_2 O_i} c_{Q_3 Q_4 O_i}}{N_i} . \tag{5.5}$$

P_n is what results from the explicit perturbative calculations, after the contributions of the leading operators have been removed. Here we shall sketch the procedure one has to follow in order to compute P_n in the case of interest (scalars of naive dimension $\Delta_0 = 4$). The generalisations to higher Δ_0 and to tensor operators is straightforward, though algebraically rather involved. One first expands in double power series for small x_{12} and x_{34} the correlator

$$(-1)^n n! \langle Q_1(x_1) Q_2(x_2) Q_3(x_3) Q_4(x_4) \rangle \Big|_{g^{2n}} , \tag{5.6}$$

keeping only the terms proportional to $\ln^n(x_{24}^2)$. As already noted, consideration of the other leading logarithmic terms, proportional to $\ln^n(x_{12}^2)$ and $\ln^n(x_{34}^2)$, would lead to equivalent conclusions, but the equations coming from the $\ln^n(x_{24}^2)$ terms are simpler, since they are manifestly independent of the anomalous dimensions of the external operators. The result of the expansion has (in general) power singularities for small x_{12} and x_{34}. These come from intermediate scalar operators of naive dimension $\Delta_0^S = 2$, from vectors of naive dimension $\Delta_0^V = 3$ and from symmetric rank two traceless tensors of naive dimension $\Delta_0^T = 4$. In order to single out the contribution of the scalars of dimension $\Delta_0 = 4$ we are interested in, one has to subtract the contributions of all the above operators, together with their derivatives (descendants). This procedure is made possible by the fact that the coefficients with which descendants contribute are completely determined by conformal invariance. Notice that contributions coming from the second derivative of a scalar, as well as those coming from the first derivative of a vector or the trace of a tensor give rise to regular behaviours. Consequently, their subtraction is crucial to get the correct value of P_n, which is the residual coefficient of the regular term we are after.

The generalization of eqs. (5.2) to (5.4) to order $\mathrm{O}(g^{2n})$ is straightforward and reads

$$\sum_{i=1}^{k} F_i(N)(\gamma_i(N))^n = P_n(N)\,, \tag{5.7}$$

where we made explicit the dependence of both F_i and P_n on the number of colours N. Note that the N dependence of P_n for $n \leq 3$ is polynomial for all four-point functions we computed. Indeed the N dependence of the four-point functions comes only from the traces over the colour indices which can be rewritten as traces of products of an even number of $\mathrm{SU}(N)$ matrices in the adjoint representation. For $\ell = 2, 4, 6$ matrices only the "planar" factor $(N)^\ell(N^2-1)$ appears. For more than eight matrices, non-planar contributions start to appear with colour factors that will depend on the exact structure of the four-point function under consideration.

Eliminating the quantities $F_i(N)$ from the system (5.7) leads to the following consistency equations

$$P_{k+L} - \left(\sum_i \gamma_i\right) P_{k+L-1} + \left(\sum_{i<j} \gamma_i\gamma_j\right) P_{k+L-2} - \left(\sum_{i<j<l} \gamma_i\gamma_j\gamma_l\right) P_{k+L-3} + \cdots$$
$$+(-1)^k \gamma_1\gamma_2\ldots\gamma_k P_L = 0 \tag{5.8}$$

for any $L \geq 0$. Eqs. (5.8) imply that the combinations of anomalous dimensions

$$\sum_i \gamma_i\,, \qquad \sum_{i<j} \gamma_i\gamma_j\,, \qquad \sum_{i<j<l} \gamma_i\gamma_j\gamma_l\,, \qquad \ldots \qquad \gamma_1\gamma_2\ldots\gamma_k \tag{5.9}$$

as well as all the P_n's are completely determined, once one knows the leading logarithmic behaviour up to order $\mathrm{O}((g^2)^{2k-1})$. In fact to solve for the k variables (5.9) one needs eq. (5.8) for k different values of L. Thus the knowledge of the coefficients P_n from $n = 0$ to $n = 2k - 1$ is enough to determine everything. Unfortunately, except for some special cases, the currently available data is far from meeting the minimal information required, even when k is relatively small (e.g. $k = 3$ or $k = 4$). Considering several different correlators only partially improves the situation, since the number of unknowns is anyway very large.

On the other hand the knowledge of only P_0, P_1 and P_2 allows one to obtain bounds for the smallest and largest anomalous dimensions of the operators contributing to the four-point functions of two identical pairs of operators of the form $\langle Q_1 Q_2 Q_1 Q_2 \rangle$, even without knowing the explicit expressions of the intermediate operators O_i.

Indeed if $\gamma_{\rm min}$ is the smallest anomalous dimension (or any of them if there are several), then multiplying eq. (5.2) by $\gamma_{\rm min}$ and subtracting it from eq. (5.3), one obtains

$$\sum_i F_i(\gamma_i - \gamma_{\rm min}) = (P_1 - \gamma_{\rm min} P_0)\,. \tag{5.10}$$

The left hand side is non-negative, since by assumption $\gamma_{\rm min}$ is smaller than or equal to all the other γ_i, while F_i are non-negative due to the assumption $Q_3 = Q_1$, $Q_4 = Q_2$. One thus derives the inequality

$$\gamma_{\rm min} \leq \frac{P_1}{P_0}\,. \tag{5.11}$$

Calling $\gamma_{\rm Max}$ the largest of the anomalous dimension and repeating the previous argument, we find

$$\gamma_{\rm Max} \geq \frac{P_1}{P_0}\,. \tag{5.12}$$

Both in (5.11) and in (5.12) the equality is reached if all the operators have the same anomalous dimension.

In the same way, if $\gamma_{\rm s}$ is the anomalous dimension with smallest square (or any of them if there are several), then multiplying eq. (5.2) by $\gamma_{\rm s}^2$ and subtracting it from eq. (5.4), one gets

$$\sum_i F_i((\gamma_i)^2 - (\gamma_{\rm s})^2) = (P_2 - (\gamma_{\rm s})^2 P_0)\,, \tag{5.13}$$

implying

$$\gamma_{\rm s}^2 \leq \frac{P_2}{P_0}\,. \tag{5.14}$$

An important implication of these considerations is that, if $P_2 = 0$ (which implies also $P_1 = 0$) in some Green function, then the $\mathrm{O}(g^2)$ corrections to the anomalous dimensions of all the operators that contribute to it are zero, since all the non-negative products $F_i \gamma_i^2$ vanish. Thus for each i one either has $F_i = 0$, which means that the corresponding operator is not present in the OPE, or $\gamma_i = 0$. Let us stress that the vanishing of P_1 alone cannot guarantee the vanishing of the anomalous dimensions, due to possible cancellations among positive and negative terms. However, if P_1 is negative, then it is immediate to deduce that at least one of the anomalous dimensions (and in particular the smallest one) must be negative.

6. The icosaplet and the singlet

In this section we will present the results one can get for the anomalous dimensions of the operators of naive conformal dimension $\Delta_0 = 4$ in the singlet and in the icosaplet (the representation $\mathbf{20'}$), making use of the $\mathrm{O}(g^2)$ computations reported in the previous section and of the $\mathrm{O}(g^4)$ results obtained in ref. [7, 36].

For the purpose of isolating the representations of interest it turns out to be convenient to consider the correlators

$$\langle [\mathcal{C}^{11}(x_1)\mathcal{C}^\dagger_{11}(x_2) - \mathcal{C}^{22}(x_1)\mathcal{C}^\dagger_{22}(x_2)][\mathcal{C}^{11}(x_3)\mathcal{C}^\dagger_{11}(x_4) - \mathcal{C}^{22}(x_3)\mathcal{C}^\dagger_{22}(x_4)]\rangle \tag{6.1}$$

for the study of the icosaplet and

$$\sum_{I,J=1}^{3} \langle \mathcal{C}^{II}(x_1)\mathcal{C}^\dagger_{II}(x_2)\mathcal{C}^{JJ}(x_3)\mathcal{C}^\dagger_{JJ}(x_4)\rangle \tag{6.2}$$

for the singlet. Both the above correlators in the relevant channels actually receive contributions also from $\Delta_0 = 4$ operators belonging to other representations, namely the $\mathbf{105}$ and the $\mathbf{84}$. While all the operators in the $\mathbf{105}$ are protected,[5] there exist operators in the $\mathbf{84}$ that belong to the unprotected Konishi supermultiplet. Hence a laborious subtraction procedure is required to single out the contribution of individual representations.

Let us start discussing the case of the representation $\mathbf{20'}$. There are 6 possible operators of naive dimension $\Delta_0 = 4$. Two of them, $\mathcal{K}_{\mathbf{20'}}$ and $\mathcal{K}_{\mathbf{20'}}{}^\dagger$, are of the Yukawa type at leading order in g (see eq. (3.10)) and do not contribute to functions of only scalars at the order we work. The other four operators are at leading order purely scalar (since they have to be orthogonal to $\mathcal{K}_{\mathbf{20'}}$ and $\mathcal{K}_{\mathbf{20'}}{}^\dagger$). In preparation to our later analysis, it is convenient to split them into a double trace operator

$$D^{(ij)}_{\mathbf{20'}}(x) =: \mathcal{Q}^{(ik)}_{\mathbf{20'}}(x)\mathcal{Q}^{(jk)}_{\mathbf{20'}}(x) - \frac{\delta_{ij}}{6}\mathcal{Q}^{(kl)}_{\mathbf{20'}}(x)\mathcal{Q}^{(kl)}_{\mathbf{20'}}(x) : \tag{6.3}$$

which appears in the OPE of $\mathcal{Q}^{(i_1 j_1)}_{\mathbf{20'}}(x) \cdot \mathcal{Q}^{(i_2 j_2)}_{\mathbf{20'}}(y)$ and three addittional operators, spanning a 3-dimensional space orthogonal to $D_{\mathbf{20'}}$. In this space it is convenient to take as reference "directions" the double trace operator

$$\hat{M}^{(ij)}_{\mathbf{20'}}(x) =: \mathcal{Q}^{(ij)}_{\mathbf{20'}}(x)\mathcal{K}_{\mathbf{1}}(x) : -\frac{6}{3N^2-2}D^{(ij)}_{\mathbf{20'}}(x)\,, \tag{6.4}$$

[5] The representation $\mathbf{105}$ has Dynkin labels $[0,4,0]$.

which couples to $\mathcal{Q}^{(ij)}_{\mathbf{20'}}(x) \cdot \mathcal{K}_{\mathbf{1}}(y)$ and two single trace operators that vanish for SU(2). One, $L_{\mathbf{20'}}$, is proportional to the quartic Casimir of the SU(N) gauge group, the other, $O_{\mathbf{20'}}$, is a linear combination of $D_{\mathbf{20'}}$, $\hat{M}_{\mathbf{20'}}$ and $\mathrm{tr}([\varphi^i,\varphi^k][\varphi^k,\varphi^j]) - \delta^{ij}/6\,\mathrm{tr}([\varphi^l,\varphi^k][\varphi^k,\varphi^l])$.

After subtracting out the contribution of the lowest dimensional operators, we find from eq. (6.1) $P_2 = 0$. According to our previous discussion, this means that the O(g^2) correction to the anomalous dimension of $D_{\mathbf{20'}}$, which we know has a non-zero coupling to $\mathcal{Q}_{\mathbf{20'}} \cdot \mathcal{Q}_{\mathbf{20'}}$, has to vanish. The three other scalar operators are orthogonal to $D_{\mathbf{20'}}$. Tree-level orthogonality (eq. (5.1)) is enough to ensure the absence of contributions from these operators to all logarithmic corrections (even to the subleading ones) at order O(g^4). Consequently, in agreement with [36], we conclude that also the O(g^4) correction to the anomalous dimension of $D_{\mathbf{20'}}$ is zero. The vanishing of the O(g^2) anomalous dimension was first pointed out in [16]. The absence of O(g^4) correction confirms the conjecture made by these authors that $D_{\mathbf{20'}}$ is protected, although there is no known shortening condition associated to its quantum numbers.

Since the analysis of the amplitude $\langle \mathcal{C}\mathcal{C}^\dagger \mathcal{V}\mathcal{K}_{\mathbf{1}}\rangle$ does not lead to independent relations (it merely confirms previous O(g^2) results), we are left with only one constraint from the Green function $\langle \mathcal{C}\mathcal{C}^\dagger \mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle$ (eqs. (4.26) and (4.27)). Although this is not enough to determine the six remaining unknowns (the 3 angles of mixing and the 3 anomalous dimensions of the operators $\hat{M}_{\mathbf{20'}}$, $O_{\mathbf{20'}}$ and $L_{\mathbf{20'}}$), we can, nevertheless, obtain bounds for the order O(g^2) corrections to the anomalous dimensions of the operators appearing in the OPE of $\mathcal{Q}^{(ij)}_{\mathbf{20'}}(x) \cdot \mathcal{K}_{\mathbf{1}}(y)$, namely

$$\gamma_{\min} \leq \frac{(3N^2-2)}{(N^2-2)} \times \frac{g^2 N}{4\pi^2} \leq \gamma_{\mathrm{Max}}\,. \tag{6.5}$$

From eq. (6.5) we conclude that in the limit $N \to \infty$, $g^2 N$ fixed, γ_{Max} will be non-vanishing, like what happens for $\mathcal{K}_{\mathbf{1}}$.

The case of SU(2) is peculiar, since both $O_{\mathbf{20'}}$ and $L_{\mathbf{20'}}$ vanish, hence the only relevant operator is $\hat{M}_{\mathbf{20'}}$. Its anomalous dimension can then be determined and turns out to be equal to $5 \times \frac{2g^2}{4\pi^2}$.

In the singlet sector again there are too many operators and from the computations we have at our disposal we can only get bounds for the relevant anomalous dimensions. The smallest possible value for the O(g^2) anomalous dimension of the $\Delta_0 = 4$ singlet operators contributing to the OPE of $\mathcal{Q}^{(i_1 j_1)}_{\mathbf{20'}}(x) \cdot \mathcal{Q}^{(i_2 j_2)}_{\mathbf{20'}}(y)$ turns out to be negative and to satisfy the bound

$$\gamma_{\min} \leq -\frac{12}{(3N^2-1)} \times \frac{g^2 N}{4\pi^2} \leq 0\,. \tag{6.6}$$

For the smallest square, γ_{s} (γ_{s} may be different from $\gamma_{\min}$), one finds the inequality

$$(\gamma_{\mathrm{s}})^2 \leq \frac{54}{(3N^2-1)} \times \left(\frac{g^2 N}{4\pi^2}\right)^2\,. \tag{6.7}$$

For the anomalous dimensions of the operators contributing to the $\mathcal{K}_{\mathbf{1}}(x)\cdot\mathcal{K}_{\mathbf{1}}(y)$ OPE we finally find

$$\gamma_{\min}\leq\frac{3(6N^2+1)}{3N^2-2}\times\frac{g^2N}{4\pi^2}\leq\gamma_{\mathrm{Max}}\,. \tag{6.8}$$

Again in the limit $N\to\infty$, g^2N fixed, γ_{Max} will be non-vanishing, like in the previous case. Note that two sets of operators contributing to the OPE of $\mathcal{K}_{\mathbf{1}}(x)\cdot\mathcal{K}_{\mathbf{1}}(y)$ and $\mathcal{Q}^{(i_1j_1)}_{\mathbf{20'}}(x)\cdot\mathcal{Q}^{(i_2j_2)}_{\mathbf{20'}}(y)$ have a non-trivial intersection, since the function $\langle\mathcal{C}\mathcal{C}^\dagger\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle$ is non-vanishing, but obviously do not coincide.

We conclude this section by observing that without performing further calculations, either at higher orders or involving operators with fermionic content, it is impossible to disentangle all the $\Delta_0=4$ operators belonging to the singlet and the icosaplet representations. One might be tempted to try an ansatz that satisfy all the constraints with a number of operators smaller than the maximum in principle allowed. Indeed there are many such possible choices. Rather surprisingly, it turns out that a very severe constraint on any such conjecture comes from the requirement that the right hand side of eq. (5.7) has to be polynomial in N for any $n\leq3$.

7. Vanishing of instanton contributions

We would like to prove the validity of the non-perturbative result

$$\langle\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle_{\mathrm{np}}=0 \tag{7.1}$$

for any instanton number and any gauge group.

Let us start considering the case of one-instanton ($\kappa=1$) and SU(2) gauge group. For this type of calculations it is more convenient to work in the Wess-Zumino gauge, where the operator $\mathcal{K}_{\mathbf{1}}$ takes the simple form

$$\mathcal{K}_{\mathbf{1}}=\frac{1}{4}\sum_{i=1}^{6}\sum_{a=1}^{3}\varphi^i_a\varphi^a_i\,. \tag{7.2}$$

The effective one-instanton contribution to the fundamental scalars, φ, is given by

$$\varphi^i_a=f^2(x)t^i_{AB}\zeta^A\sigma_a\zeta^B\,, \tag{7.3}$$

where

$$f(x)=\frac{\rho}{(x-x_0)^2+\rho^2} \tag{7.4}$$

is the instanton profile function (incidentally, in the AdS/CFT correspondence [10]–[12] $f(x)$ plays the rôle of boundary-to-bulk propagator [22]) and

$$\zeta^A_\alpha=\eta^A_\alpha+\hat{x}_{\alpha\dot\alpha}\bar\xi^{A\dot\alpha} \tag{7.5}$$

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is a Weyl spinor in the representation **4** of SU(4) with $\hat{x}_{\alpha\dot{\alpha}} = x_\mu \sigma^\mu_{\alpha\dot{\alpha}}$. Explicitly, inserting eqs. (7.3) and (7.5) in (7.2), we get

$$\mathcal{K}_{\mathbf{1}}\Big|_{\text{inst}} = f^4(x)\delta_{ij} t^i_{AB}(\zeta^A \sigma_a \zeta^B) t^j_{CD}(\zeta^C \sigma_a \zeta^D)\,. \tag{7.6}$$

From the completeness relation of σ matrices and the SU(4) relation

$$\delta_{ij} t^i_{AB} t^j_{CD} = 2\epsilon_{ABCD}\,, \tag{7.7}$$

one immediately gets the identity

$$\mathcal{K}_{\mathbf{1}}\Big|_{\text{inst}} = 2f^4(x)\epsilon_{ABCD}(\zeta^A\zeta^B)(\zeta^C\zeta^D) = 0\,. \tag{7.8}$$

The last equality follows from the antisymmetry of the ϵ-symbol and the symmetry of $(\zeta^A\zeta^B)$. The vanishing of the one-instanton contribution to the operator $\mathcal{K}_{\mathbf{1}}$ trivially implies for the SU(2) case

$$\langle \mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle^{\mathrm{SU}(2)}_{\kappa=1} = 0\,, \tag{7.9}$$

since there is no way to absorb the 16 exact fermionic zero-modes that exist in the one-instanton background.[6]

The generalization to higher instanton numbers and other gauge groups is as always straightforward. An instanton correlator may be non-vanishing only if all the "unlifted" 16 supersymmetric and superconformal fermionic zero-modes are absorbed by suitable operator insertions. But we have just shown that $\mathcal{K}_{\mathbf{1}}$ can be of no help in this job. Thus the correlator $\langle \mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle$ vanishes for any κ and any gauge group. The same type of argument leads to the conclusion that the three-point function $\langle \mathcal{Q}_{\mathbf{20}'}\mathcal{Q}_{\mathbf{20}'}\mathcal{K}_{\mathbf{1}}\rangle_\kappa$ and four-point correlators, such as $\langle \mathcal{Q}_{\mathbf{20}'}\mathcal{Q}_{\mathbf{20}'}\mathcal{K}_{\mathbf{1}}\mathcal{K}_{\mathbf{1}}\rangle_\kappa$ or $\langle \mathcal{Q}_{\mathbf{20}'}\mathcal{Q}_{\mathbf{20}'}\mathcal{Q}_{\mathbf{20}'}\mathcal{K}_{\mathbf{1}}\rangle_\kappa$, are zero for any κ and any gauge group. Similar results extend to other operators in the Konishi supermultiplet. For instance

$$\langle \mathcal{K}(x_1)\ldots\mathcal{K}(x_n)\rangle_\kappa = 0 \tag{7.10}$$

for any $n \leq 4$, any κ and any choice of $\mathcal{K}$ in the multiplet.

However, five- and higher-point functions with insertions of operators in the Konishi multiplet can receive non-vanishing contributions from instantons. The simplest example of a correlator of this type is

$$\langle \mathcal{Q}_{\mathbf{20}'}(x_1)\mathcal{Q}_{\mathbf{20}'}(x_2)\mathcal{Q}_{\mathbf{20}'}(x_3)\mathcal{Q}_{\mathbf{20}'}(x_4)\mathcal{K}_{\mathbf{1}}(x_5)\rangle\,. \tag{7.11}$$

Similarly the simplest non-vanishing correlator with only $\mathcal{K}_{\mathbf{1}}$ insertions is the eight-point function.

[6] We recall that the 16 (supersymmetric and superconformal) fermionic zero-modes, generated by the 16 possible independent choices of the spinors η^A and $\bar{\xi}^A$ in eq. (7.5), are lifted neither by the Yukawa couplings nor by the quartic potential term of the scalars [24].

Based on the OPE analysis of the one-instanton contribution to the four-point correlator $\langle \mathcal{Q}_{\mathbf{20'}} \mathcal{Q}_{\mathbf{20'}} \mathcal{Q}_{\mathbf{20'}} \mathcal{Q}_{\mathbf{20'}} \rangle$, we proved in ref. [6] the result $\gamma_{\mathcal{K}_\mathbf{1}}^{(\kappa=1)} = 0$. The new calculations presented here seem to point to a much larger "inertia" of the Konishi operators to non-perturbative corrections, namely to the vanishing of the whole non-perturbative correction to their anomalous dimension ($\gamma_{\mathcal{K}_\mathbf{1}}^{(\mathrm{np})} \equiv 0$) and to the vanishing of similar corrections to trilinear couplings, like $\langle \mathcal{K}_\mathbf{1} \mathcal{K}_\mathbf{1} \mathcal{K}_\mathbf{1} \rangle$ or $\langle \mathcal{Q}_{\mathbf{20'}} \mathcal{Q}_{\mathbf{20'}} \mathcal{K}_\mathbf{1} \rangle$.

8. Conclusions

In this concluding section we would like to briefly summarize the present understanding of the spectrum and properties of composite operators in $\mathcal{N} = 4$ SYM in view of the AdS/CFT correspondence.

First of all, there are short multiplets with maximal spin 2, which are dual to the supergravity multiplet and its Kaluza-Klein (KK) excitations. Their scaling dimensions, which are integer for bosons and half-integer for fermions, do not receive quantum corrections and the same seems to be true for their trilinear couplings [25, 26, 6, 7]. AdS computations at strong coupling [27] have shown that extremal [28, 29] and next-to-extremal [30] correlators of these operators do not receive quantum corrections either. The AdS results have been tested both in perturbation theory and non-perturbatively with $\mathcal{N} = 4$ field theoretical computations [30]–[32].

Other short multiplets with spin larger than two, that cannot possibly be dual to the supergravity fields or their KK excitations, are known to have similar non-renormalization properties [33]. Multitrace operators of this kind are interpreted as dual to BPS-like bound states at threshold.

Next there are multitrace operators in long multiplets with anomalous dimensions that vanish in the large N limit and are thus visible in the supergravity approximation [34, 35]. Their interpretation is more troublesome. They can be viewed as non-BPS bound states, since their anomalous dimensions are of the correct order of magnitude, i.e. $\gamma \propto 1/N^2$, to be holographically dual to gravitational binding energies,[7] but it is not clear what is the meaning that should be attached to a "classical" bound state.[8]

We then have long Konishi-like multiplets that are expected to acquire large anomalous dimensions ($\Delta \approx (g^2 N)^{1/4}$) and decouple from the operator algebra at strong 't Hooft coupling in the large N limit. They are dual to string excitations, whose mass is of the order of Δ in AdS units. In this paper we have confirmed previous results on the perturbative anomalous dimension of the $\mathcal{N} = 4$ Konishi multiplet and gone one step further on the non-perturbative side by showing the vanishing of instanton contributions to correlators with up to four operator insertions.

[7] A gravitational-like mass defect $\delta M = G_N M^2/L^2 \approx g_s^2/L = g_{YM}^4 N^2/N^2 L$ precisely corresponds to an anomalous dimension $\gamma \propto 1/N^2$ at fixed 't Hooft coupling.

[8] We would like to thank A.C. Petkou for an interesting discussion on this point.

Finally there are unprotected operators that rather surprisingly do not receive corrections to their tree-level dimensions. This was known for a double trace operator of dimension $\Delta_0 = 4$ belonging to the representation $\mathbf{20'}$ of SU(4) at $\mathrm{O}(g^2)$ and including one-instanton corrections [16]. We have confirmed this at $\mathrm{O}(g^4)$ (see also [36]) and for any instanton number in the present paper. Partial non-renormalization of near-extremal correlators of CPO's [37], and in particular a functional relation between two *a priori* independent four-point functions of lowest dimension CPO's [7, 23], seems to be at the heart of the vanishing of the anomalous dimensions of some unprotected operators [36]. In this context an important step forward has been made in ref. [36], where the constraints imposed by $\mathcal{N} = 4$ superconformal invariance on four-point functions have been studied along the lines of what has been done in [38].

We would like to conclude with a few comments on the $\mathrm{U}_B(1)$ bonus symmetry of ref. [21] and on the $\mathrm{SL}(2,\mathbb{Z})$ invariance expected to be realized in $\mathcal{N} = 4$ SYM.

On the first issue our results confirm the conjecture that correlators of single-trace operators in short multiplets with up to four-point as well as three-point functions with at most one operator belonging to a long multiplet obey the $\mathrm{U}_B(1)$ selection rule.

As for the question of how $\mathrm{SL}(2,\mathbb{Z})$ invariance is realized, we can make the following observations. Since physics is θ dependent in $\mathcal{N} = 4$ SYM, we expect observables, such as anomalous dimensions and trilinear couplings, to generally depend on the vacuum angle. It is thus conceivable that some of the observables may be non-holomorphic, but still θ dependent, modular functions of the complexified coupling, $\tau = \theta/2\pi + 4\pi i/g^2$. The class of operators whose dimensions and couplings may depend on τ in a modular invariant way is, however, restricted to the operators with $\gamma \propto 1/N^2$ in the large N limit, like, for instance, the unprotected double-trace operators of dimension 4 in the singlet of SU(4). As remarked above, when discussing the relation between gravitational binding energy of non BPS states and anomalous dimensions, this kind of non-BPS multi-trace operators (dual to non BPS multi-particle states) are rather "elusive" from the AdS perspective. Konishi-like operators, on the contrary, do not seem to receive any non-perturbative corrections to their anomalous dimensions and trilinear couplings. This means that none of the observable quantities associated to them can possibly show simple modular properties. A better understanding of string theory on AdS space and in general on backgrounds with non-vanishing RR charge might shed some light on the pattern of dimensions of operators dual to string excitations and non-BPS bound states of KK excitations.

In conclusion $\mathcal{N} = 4$ SYM seems to be a very interesting theory. Though superconformal symmetry strongly constrains the dynamics, it allows for interesting features to emerge both at weak coupling (in perturbation theory and non-perturbatively) and at strong coupling, where the supergravity description is in good qualitative agreement with field theory expectations [39].

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Some puzzling features call for a deeper understanding of the rôle of superconformal invariance on the dynamics or for the emergence of some new hidden symmetries, such as the $\mathrm{U}_B(1)$ bonus symmetry or symmetries associated to "central" extensions of the superconformal algebra.

Acknowledgments

We would like to acknowledge stimulating discussions with B. Eden, S. Ferrara, D. Freedman, M.B. Green, P. Howe, H. Osborn, A.C. Petkou, A. Sagnotti, K. Skenderis, E. Sokatchev and G. Veneziano. This work was partly supported by the EEC contract HPRN-CT-2000-00122. M.B. also ackowledges partial financial support by PPARC and the INTAS project 991590.

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